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Generalized coherent states
for q-oscillator connected with
q-Hermite Polynomials [1]

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Abstract

For the oscillator-like systems, connected with q-Hermite polynomials, coherent states of Barut-Girardello type are defined. The well-known Arik-Coon oscillator naturally arose in the framework of suggested approach as oscillator, connected with the Rogers q-Hermite polynomials, in the same way as usual oscillator with standard Hermite polynomials. The results about the coherent states for discrete q-Hermite polynomials of II type are quite new.

1 Introduction

We consider the oscillator-like systems (or generalized oscillators), connected with q-Hermite polynomials in the same way as usual boson oscillator connected with the standard Hermite polynomials. We define the analogues of coherent states of Barut-Girardello type for these generalized oscillators. One year ago we talked here about GCS connected with classical polynomials. Our approach to such construction is developed in the following papers [1] - [8].

Within of Askey - Wilson scheme [9], [10], we know q-Hermite polynomials of three kind only

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1. The **Rogers q-Hermite polynomials**

$$H_n(x;q) = \sum_{k=1}^{n} \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}} \exp\left\{i(n-2k)\theta\right\}, \tag{1}$$

   where $x = \cos\theta$, $0 < q < 1$ and

$$(q;q)_k = \prod_{s=1}^{k}(1-aq^{s-1}), \quad (q;q)_\infty = \prod_{s=1}^{\infty}(1-aq^{s-1}). \tag{2}$$

2. The **discrete q-Hermite polynomials of I-type**

$$h_n(x;q) = q^{\binom{n}{q}} {}_2\phi_1\left(\left.\begin{matrix} q^{-n},\, x^{-1} \\ 0 \end{matrix}\right| q;\, -qx\right) \tag{3}$$

3. The **discrete q-Hermite polynomials of II-type**

$$\tilde{h}_n(x;q) = x^n {}_2\phi_1\left(\left.\begin{matrix} q^{-n},\, q^{-n+1} \\ 0 \end{matrix}\right| q^2;\, -x^{-2}\right). \tag{4}$$

In the first case we get the well-known q-oscillator of Arik - Coon [11]. We want to stress that in this case we obtain these well-known results using a new approach.

In the second case, it turned out, that no GCS exist. This is due to the fact, that the radius of convergence for normalizing factor of GCS equals zero.

Finally, in the third case we will construct GCS. These results are quite new. Unfortunately, we do not know the measure involved in "resolution of unity". So the completeness of GCS has not yet been established.

The main results are

- We show that our approach to construction of coherent states works in the deformed case as well as in the classical one.
- The well-known Arik -Coon oscillator naturally arose in our approach as oscillator connected with the q-Hermite polynomials in the same way as usual oscillator with standard Hermite polynomials.

2 Coherent states for Rogers q-Hermite polynomials $H_n(x;q)$ and Arik - Coon oscillator

2.1 q-Oscillator, connected with $H_n(x;q)$

In the Hilbert space

$$\mathcal{H}_q = \mathrm{L}^2\left([-1,1];\, \mathrm{d}\mu_q(x)\right), \qquad \mathrm{d}\mu_q(x) = \frac{(q;q)_\infty}{2\pi} \frac{\left|(e^{2i\theta};q)_\infty\right|^2}{\sqrt{1-x^2}}, \tag{5}$$

we consider the orthonormal basis given by Rogers q-Hermite polynomials

$$\varphi_n(x;q) = (q;q)_n^{-1/2} H_n(x;q). \tag{6}$$

The recurrence relations for the polynomials $\varphi_n(x;q)$ have the form

$$x\varphi_n(x;q) = b_n\varphi_{n+1}(x;q) + b_{n-1}\varphi_{n-1}(x;q), \qquad \varphi_0(x;q) = 1, \tag{7}$$

with coefficients

$$b_n = \frac{1}{2}\sqrt{1-q^{n+1}}, \quad n \geq 0, \qquad b_{-1} = 0. \tag{8}$$

We consider $\mathcal{H}_q$ as the Fock space for the deformed oscillator defined by the following operators ($|n\rangle := \varphi_n(x;q)$)

$$X_q|n\rangle = b_n|n+1\rangle + b_{n-1}|n-1\rangle; \tag{9}$$

$$P_q|n\rangle = i\left(b_n|n+1\rangle - b_{n-1}|n-1\rangle\right); \tag{10}$$

$$a_q^+ = (1-q)^{-\frac{1}{2}}\left(X_q - iP_q\right), \quad a_q^+|n\rangle = \sqrt{[n+1]_q}|n+1\rangle; \tag{11}$$

$$a_q^- = (1-q)^{-\frac{1}{2}}\left(X_q + iP_q\right), \quad a_q^-|n\rangle = \sqrt{[n]_q}|n-1\rangle, \tag{12}$$

where

$$[n]_q = \frac{1-q^n}{1-q}$$

is the "mathematical" q-number and we consider the case $0 < q < 1$. From above relations we obtain

$$a_q^- a_q^+ - q a_q^+ a_q^- = 1, \tag{13}$$

which means that we obtain the Arik - Coon oscillator.

The polynomials $|n\rangle = \varphi_n(x;q)$ are eigenfunctions of the Hamiltonian

$$\begin{gathered} H_q = {X_q}^2 + {P_q}^2 = a_q^+ a_q^- + a_q^- a_q^+, \\ H_q|n\rangle = \lambda_n|n\rangle; \\ \lambda_0 = \frac{4b_0^2}{1-q} = 1; \quad \lambda_n = \frac{4}{1-q}\left(b_{n-1}^2 + b_n^2\right) = \left([n]_q + [n+1]_q\right) \end{gathered} \tag{14}$$

The equation (14) is equivalent to q-difference equation for the Rogers q-Hermite polinomials

$$(1-q)D_q\left[w(x)D_q\varphi_n(x;q)\right] + 4q^{1-n}[n]_q w(x)\varphi_n(x;q) = 0, \tag{15}$$

where

$$w(x) = \left|(e^{2i\theta};q)_\infty\right|^2, \qquad D_q f(x) := \frac{\delta_q f(x)}{\delta_q x}, \tag{16}$$

with

$$\delta_q f(e^{i\theta}) = f(q^{\frac{1}{2}}e^{i\theta}) - f(q^{-\frac{1}{2}}e^{i\theta}), \quad x = \cos\theta. \tag{17}$$

The unitary equivalence of the position and the momentum operators is given by a generalization of the Fourier transform. Besides, the Hamiltonian (14) is invariant under generalized Fourier transform as well as in the classical case.

Let's note that from the other point of view the connection of Arik - Coon oscillator with Rogers q-Hermite polynomials was mentioned in [12, 13].

2.2 Generalized Fourier transform for Rogers q-Hermite polynomials

We define Generalized Fourier transform (GFT) related to the orthonormal system $\{\varphi_n(x;q)\}_0^\infty$ by the relation

$$\mathrm{F}_\varphi f(y) = \int_{-1}^{1} \mathrm{K}_\varphi(x,y;-i) f(x)\mu_q(x), \tag{18}$$

where the Poisson kernel is given by

$$\mathrm{K}_\varphi(x,y;t) = \sum_{n=0}^{\infty} t^n \varphi_n(x;q)\varphi_n(y;q) \quad \text{and} \quad \mathrm{K}_\varphi(x,y;-i) = \lim_{t\to -i} \mathrm{K}_\varphi(x,y;t).$$

From q-Mehler formula for Rogers q-Hermite polynomials $H_n(x;q)$ we obtain

$$\mathrm{K}_\varphi(x,y;t) = \frac{(t^2;q)_\infty}{|(te^{2i\theta};q)_\infty (t;q)_\infty|^2}. \tag{19}$$

It is easily to check, that our definition of GFT conforms with given (from the different considerations) in [14].

2.3 Coherent states for the q-oscillator, connected with Rogers q-Hermite polynomials $H_n(x;q)$

The Barut - Girardello coherent states are defined by

$$a_q^- |z\rangle = z|z\rangle; \qquad |z\rangle = \mathcal{N}^{-1} \sum_{n=0}^{\infty} \frac{H_n(x;q)}{\sqrt{(q;q)_n}} \frac{z^n}{\sqrt{[n]_q!}}, \tag{20}$$

where

$$\mathcal{N}^2 = \sum_{n=0}^{\infty} \frac{|z|^{2n}}{[n]_q!} = \tilde{e}_q((1-q)|z|^2); \qquad R = \frac{1}{\sqrt{1-q}}. \tag{21}$$

Here R is the radius of convergence and by definition

$$\tilde{e}_q(x) := \sum_{n=0}^{\infty} \frac{x^n}{(q;q)_n}. \tag{22}$$

(Let's note that we reserve the notation $e_q(x)$ for slightly different definition of q-exponential function

$$e_q(x) := \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!} = \tilde{e}_q((1-q)x), \tag{23}$$

which is more customary for quantum groups theory.)

Using the generation function for the Rogers q-Hermite polinomials, from (20) we obtain

$$|z\rangle = \frac{\tilde{e}_q(e^{i\theta}\sqrt{1-q}z)\tilde{e}_q(e^{-i\theta}\sqrt{1-q}z)}{\sqrt{\tilde{e}_q((1-q)|z|^2)}}. \tag{24}$$

The overlap of two coherent states is given by

$$\langle z_1|z_2\rangle_q = \tilde{e}_q((1-q)\bar{z}_1 z_2). \tag{25}$$

To prove "resolution of identity" property

$$\int\int_{\mathbb{C}_{1/(1-q)}} |z\rangle_q\langle z|\mathrm{d}\mu(|z|^2) = I, \quad \mathrm{d}\mu(|z|^2) = W(|z|^2)\mathrm{d}(\mathrm{Re}z)\mathrm{d}(\mathrm{Im}\ z), \tag{26}$$

which means that the set of coherent states is complete (really, over complete) we must solve the moment problem

$$\int_0^{1/(1-q)} t^n \frac{W(t)}{\mathcal{N}^2}\mathrm{d}\,t = \frac{[n]_q!}{\pi}. \tag{27}$$

Its solution is given by the distribution

$$W(t) = \frac{1-q}{\pi}\sum_{k=0}^{\infty}\frac{\mathcal{N}^2 t}{e_q(qt)}\delta\left(t - \frac{q^k}{1-q}\right), \tag{28}$$

which can be obtained from the relation ([11]):

$$I_n(q) = \int_0^{1/(1-q)} {e_q}^{-1}(qx)\,x^n \mathrm{d}_q\,x = [n]_q!, \tag{29}$$

where $I_n(q)$ is the so-called Jackson integral

$$\int_0^a f(x)\mathrm{d}_q\,x := a(1-q)\sum_{k=0}^{\infty} q^k f(q^k a). \tag{30}$$

Thus the measure in (26) is given by

$$\begin{aligned}\mathrm{d}\mu(|z|^2) &= \frac{1-q}{\pi}\sum_{k=0}^{\infty}\frac{\mathcal{N}^2|z|^2}{e_q(q|z|^2)}\delta\left(|z|^2 - \frac{q^k}{1-q}\right)\mathrm{d}^2 z \\ &= \frac{1-q}{\pi}\sum_{k=0}^{\infty}\frac{|z|^2 e_q(|z|^2)}{e_q(q|z|^2)}\delta\left(|z|^2 - \frac{q^k}{1-q}\right)\mathrm{d}^2 z. \end{aligned} \tag{31}$$

For the completeness we give the simple proof of the essential relation (29). It is well known that for q-derivative defined as

$$\left(\tfrac{\mathrm{d}}{\mathrm{d}x}\right)_q f(x) = \frac{f(x) - f(qx)}{x(1-q)}, \tag{32}$$

one has (and easily verified)

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)_q x^n = [n]_q x^{n-1}\,, \qquad\qquad \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)_q e_q(x) = e_q(x)\,. \tag{33}$$

From the Leibnitz relation for q-derivative

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)_q (u(x)v(x)) = u(x)\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)_q v(x) + v(qx)\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)_q u(x)\,, \tag{34}$$

it follows the integration by parts rule for the Jackson integral in the form

$$\int_0^a u(x)\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)_q v(x)\mathrm{d}_q\, x = \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)_q (u(x)v(x))\Big|_0^a - \int_0^a v(qx)\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)_q u(x)\mathrm{d}_q\, x. \tag{35}$$

By differentiating the identity $e_q(x)\cdot e_q^{-1}(x) = 1$ one obtains

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)_q e_q^{-1}(x) = -e_q^{-1}(qx)\,. \tag{36}$$

Using this relation and integrating by parts the Jackson integral $I_n(q)$ (29) we have

$$\begin{aligned} I_n(q) &= \int_0^{\frac{1}{1-q}} x^n \left(-\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)_q e_q^{-1}(x)\right)\mathrm{d}_q\, x = \\ &= x^n\left(-\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)_q e_q^{-1}(x)\right)\Big|_0^{\frac{1}{1-q}} + \int_0^{\frac{1}{1-q}} e_q^{-1}(qx)\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)_q x^n \mathrm{d}_q\, x\,. \end{aligned} \tag{37}$$

In view of the relation $e_q^{-1}(\frac{1}{1-q}) = 0$ one gets from (37) the recurrent relation

$$I_n(q) = [n]_q\, I_{n-1}(q)\,, \quad n \geq 1\,, \tag{38}$$

so that

$$I_n(q) = [n]_q!\, I_0(q)\,. \tag{39}$$

which gives the relation (29), because

$$I_0(q) = \int_0^{\frac{1}{1-q}} e_q^{-1}(qx)\mathrm{d}_q\, x = -e_q^{-1}(x)\Big|_0^{\frac{1}{1-q}} = e_q^{-1}(0) = 1. \tag{40}$$

Let us note that over completeness of this coherent states was considered from another point of view, for example in [15].

3 Coherent states for discrete q-Hermite polynomials II

3.1 Discrete q-Hermite polynomials II

The discrete q-Hermite polynomials of II-type are defined by

$$\tilde{h}_n(x;q) = x^n\, {}_2\phi_1\left(\left.\begin{matrix} q^{-n},\, q^{-n+1} \\ 0 \end{matrix}\,\right| q^2;\, -x^{-2}\right), \tag{41}$$

where

$$ {}_2\phi_1\left(\begin{matrix} a,\, b \\ c \end{matrix}\middle| q; z\right) = \sum_{k=0}^{\infty} \frac{(a;q)_k (b;q)_k}{(c;q)_k} \frac{z^k}{(q;q)_k}. \tag{42} $$

These polynomials fulfill the following orthogonality condition

$$ c(1-q) \sum_{k=-\infty}^{\infty} \Big[\tilde{h}_m(cq^k;q)\tilde{h}_n(cq^k;q) + \tilde{h}_m(-cq^k;q)\tilde{h}_n(-cq^k;q)\Big] w(cq^k) q^k = 0, \tag{43} $$

where $m \neq n$, $w(x) = [(ix;q)_\infty(-ix;q)_\infty]^{-1}$ and $c > 0$. We denote by $\mathcal{H}$ the Hilbert space, spanned by the set of discrete q-Hermite polynomials of II-type. The orthonormal polynomials

$$ \psi_n(x;q) = \tilde{h}_n(x;q) q^{\frac{1}{2}n^2} (q;q)_n{}^{-\frac{1}{2}}, \tag{44} $$

form the basis in $\mathcal{H}$ and fulfill the recurrence relation

$$ x\psi_n(x;q) = b_n \psi_{n+1}(x;q) + b_{n-1}\psi_{n-1}(x;q); \qquad \psi_0(x;q) = 1, \tag{45} $$

where

$$ b_n = q^{-n-\frac{1}{2}} \sqrt{1-q^{n+1}}, \quad n \geq 0; \qquad b_{-1} = 0. \tag{46} $$

3.2 Deformed oscillator connected with discrete q-Hermite polynomials II

We consider $\mathcal{H}_q$ as the Fock space for a deformed oscillator defined by the following operators ($|n\rangle := \psi_n(x;q)$)

$$ X_q|n\rangle = b_n|n+1\rangle + b_{n-1}|n-1\rangle; \tag{47} $$

$$ P_q|n\rangle = i\left(b_n|n+1\rangle - b_{n-1}|n-1\rangle\right); \tag{48} $$

$$ a_q^+ = \frac{1}{2}\sqrt{\frac{q}{1-q}}\left(X_q - iP_q\right), \quad a_q^+|n\rangle = \sqrt{\frac{q}{1-q}} b_n|n+1\rangle; \tag{49} $$

$$ a_q^- = \frac{1}{2}\sqrt{\frac{q}{1-q}}\left(X_q + iP_q\right), \quad a_q^-|n\rangle = \sqrt{\frac{q}{1-q}} b_{n-1}|n-1\rangle, \tag{50} $$

$$ N|n\rangle = n|n\rangle; \quad a_q^- a_q^+ = q^{-2N}[N+I]_q; \quad a_q^+ a_q^- = q^{-2N+2I}[N]_q; \tag{51} $$

where $[n]_q = \frac{1-q^n}{1-q}$ is the "mathematical" q-number. From the above relations we obtain

$$ a_q^- a_q^+ - q^{-1} a_q^+ a_q^- = q^{-2N}, \quad \text{or} \quad a_q^- a_q^+ - q^{-2} a_q^+ a_q^- = q^{-N}. \tag{52} $$

The polynomials $|n\rangle = \varphi_n(x;q)$ are eigenfunctions of the Hamiltonian

$$ H_q = \frac{1}{2}\frac{q}{1-q}\left(X_q{}^2 + P_q{}^2\right) = a_q^+ a_q^- + a_q^- a_q^+, \tag{53} $$

$$ H_q|n\rangle = \lambda_n|n\rangle; \tag{54} $$

$$ \lambda_n = q^{-2n}[n+1]_q + q^{2-2n}[n]_q, \quad n \geq 0. \tag{55} $$

The equation (54) is equivalent to q-difference equation for the discrete q-Hermite polynomials II

$$-(1-q^n)x^2\tilde{h}_n(x;q) = \tag{56}$$
$$= q\tilde{h}_n(x-i;q) - (1+q+x^2)\tilde{h}_n(x;q) + (1+x^2)\tilde{h}_n(x+i;q). \tag{57}$$

3.3 Coherent states for the q-oscillator, connected with $\tilde{h}_n(x;q)$

The Barut - Girardello coherent states are defined by

$$a_q^-|z\rangle = z|z\rangle; \quad |z\rangle = \mathcal{N}^{-1}\sum_{n=0}^{\infty}(q(1-q))^{\frac{1}{2}n}\, q^{n^2-n}\tilde{h}_n(x;q)\,\frac{z^n}{(q;q)_n}, \tag{58}$$

where

$$\mathcal{N}^2 = \sum_{n=0}^{\infty}\left(\frac{1-q}{q}\right)^n q^{n^2}\frac{z^{2n}}{(q;q)_n} \equiv \epsilon_b(z^2); \qquad R=\infty. \tag{59}$$

Using the generation function for the for the discrete q-Hermite polynomials II, from ([10]) we obtain

$$|z\rangle = \frac{1}{\epsilon_b(z^2)}\left(i\sqrt{q(1-q)}z;q\right)_\infty {}_1\phi_1\left(\begin{matrix} ix \\ i\sqrt{q(1-q)}z\end{matrix}\middle| q; -i\sqrt{q(1-q)}z\right), \tag{60}$$

where

$${}_1\phi_1\left(\begin{matrix} a \\ b\end{matrix}\middle| q;z\right) = \sum_{k=0}^{\infty}\frac{(a;q)_k}{(b;q)_k}\,\frac{z^k}{(q;q)_k}. \tag{61}$$

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